Optimal Broadcasting of Mixed States

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The N to M ($M \geq N$) universal quantum broadcasting of mixed states $\rho^{\otimes N}$ are proposed for qubits system. The broadcasting of mixed states is universal and optimal in the sense that the shrinking factor is independent of input state and achieves the upper bound. The quantum broadcasting of mixed qubits is a generalization of the universal quantum cloning machine for identical pure input states. A new pure state decompositions of the identical mixed qubits $\rho^{\otimes N}$ are obtained.

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I. INTRODUCTION

An unknown quantum state can not be cloned perfectly, i.e. the no-cloning theorem[1, 2], which is one fundamental theorem of quantum mechanics and quantum information. However, it does not mean that we cannot clone quantum states imperfectly or probabilistically. So far, much effort has been devoted into imperfect cloning and probabilistical cloning of quantum states[3]-[14].

The quantum cloning of pure states has already been well studied, for reviews, see Refs.[15–17]. However, the study of cloning of identical mixed states has only recently attracted some attentions [18, 20–22]. In a review paper in Rev. Mod. Phys. [15], the first open question raised by the authors is the quantum cloning of mixed states. In this paper we will study this problem with input being the identical mixed qubits. To avoid confusion, we use "broadcasting" instead of "cloning" for copying of mixed states since the output state of the processing is in general not factorized.

For pure state input without a prior information, we can construct the universal quantum cloning machine (UQCM), i.e., the quality of its output does not depend on the input. We can use distance like parameters between input and output to quantify the merit of the quantum cloning machine. For example, the Hilbert-Schmidt norm was used in Ref.[3] and later the fidelity is well accepted for various quantum cloning machines [7, 9, 15]. The quantification of the merit of the quantum broadcasting of mixed states seems more complicated. It is recently pointed out that the mixed state input cannot be universally broadcasted if we use the fidelity to describe the merit of the broadcasting [23]. It is shown that for N to M ($N \geq 2$) quantum broadcasting, where N identical input states are quantum broadcasted to M states, the fidelity between single qubit mixed input and output is input dependent unless the input is pure or completely mixed. However, the shrinking factor was used as the parameter to quantify the merit of the broadcasting of identical mixed qubits [22]. In this paper, for the reasons as the following, we will still use the shrinking factor to quantify how well the mixed qubits are copied. For *universal* quantum broadcasting, the single qubit output state generally should satisfy the scalar relation $\rho_{out}^{single} = \eta\rho + \frac{1-\eta}{2}I$ which will appear later (10), where ρ is one of the identical mixed input qubits, I is the identity operator in two-dimensional space, and η is what we called shrinking factor. We can find that the output state is actually a mixed state constructed by the original input with probability η and the completely mixed state $I/2$ with probability $1-\eta$. In quantum information processing, it is well accepted that the completely mixed state contains no information. Thus all information of the copies ρ_{out}^{single} is in state ρ. And we then assume that the shrinking factor η can be used as one parameter of the merit of the broadcasting of mixed states. It can also be understood as that the quantum state ρ_{out}^{single} is the output of a quantum depolarization channel with input ρ, where $1-\eta$ is related with the noise of this channel.

The optimal fidelity of N to M UQCM for pure state has already been obtained in Refs.[6, 7, 9]. The optimal shrinking factor can be obtained directly from the optimal fidelity as $\eta(N,M) = N(M+2)/M(N+2)$. In this paper, we shall study the quantum broadcasting with input as identical mixed qubits. We emphasize that the case considered in this paper is the *universal quantum broadcasting which is in the sense that the shrinking factor is independent from the input state. Since we know that pure states set is a subset of that of mixed states, and the optimal shrinking factor for mixed states (also including pure state case) will be upper bounded by the optimal shrinking factor for pure states.* Thus to show a broadcasting of mixed states is optimal, we only need to show that the shrinking factor is optimal. In this *universal* sense, we mean that this quantum broadcasting procession can copy mixed qubits equally well as it copies pure qubits.

Cirac *et al* [18] introduced a decomposition of the multi-qubit states of the form $\rho^{\otimes N}$, where $\rho = c_1|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|$, $c_1 + c_0 = 1$, and employed it to construct the optimal single qubit purification procedure. The same decomposition also was used to study the super-broadcasting of mixed states in Ref.[20, 21]. The states purification is also studied in Ref.[19]. In this paper, we will provide a different pure state decomposition for $\rho^{\otimes N}$, and will present an optimal *universal* quantum broadcasting for this case. For completeness, we will also present the super-broadcasting for mixed

states which was studied in Refs.[18, 20, 21]. In Ref.[22], the 2 to M mixed states broadcasting was studied in which no results about pure states decomposition of $\rho^{\otimes N}$ is involved. We study in this paper the general N to M mixed states universal broadcasting.

This paper is organized as follows. In Sec. II we introduce a new decomposition method for N identical qubits of mixed states. In Sec. III we will show how to realize the optimal universal broadcasting of mixed states from 3 to M copies by using the orthonormal states introduced in Sec. II. The quantum broadcasting transformation will be generalized to the case of $N \to M$ ($M \geq N$) mixed-states in Sec IV. In Sec. V, the super-broadcasting of mixed states will be presented. Sec. VI contains several other slightly different quantum broadcasting of mixed states. Sec. VII is a brief conclusion.

II. DECOMPOSITION METHOD FOR N IDENTICAL MIXED STATES

Before discussing the decomposition of multiqubit mixed states, we would like to review briefly the multiqubit pure states. An arbitrary pure state of two-dimensional system is described by the state vector

$$|\psi\rangle = a\,|\uparrow\rangle + b\,|\downarrow\rangle\,, \tag{1}$$

where a, b are complex numbers and $|a|^2 + |b|^2 = 1$. Therefore the N identical pure states can be written as

$$|\psi\rangle^{\otimes N} = \sum_{m=0}^{N} \sqrt{C_N^m} a^{N-m} b^m \,|(N-m)\uparrow, m\downarrow\rangle\,, \tag{2}$$

which is the superposition of all the N-qubit symmetric states, where $|(N-m)\uparrow, m\downarrow\rangle$ is a symmetric state with $N-m$ spins up and m spins down, here we use the definition $C_N^m \equiv \frac{N!}{m!(N-m)!}$. So, in the case of N identical *pure* states as input states, we only need to consider quantum cloning of symmetric states since we know $|\psi\rangle^{\otimes M}$ is still in the symmetric subspace. To ensure each single qubit of the output state is the same, we can restrict the output state of M-qubit in symmetric subspace. Actually we can show that this cloning machine can still be optimal. That is to say, the optimal universal quantum cloning of pure states can only involve quantum states in symmetric subspace.

Now, let us consider the *mixed* state input of N identical qubits, each qubit is an unknown state described by a density operator

$$\rho = c_0\,|\uparrow\rangle\langle\uparrow| + c_1\,|\downarrow\rangle\langle\downarrow|\,, \tag{3}$$

where c_0 and c_1 are the probability of the distribution and $c_1 + c_0 = 1$. Then the state of whole system, $\rho^{\otimes N}$, can be written as

$$\begin{aligned}\rho^{\otimes N} &= \left(c_0\,|\uparrow\rangle\langle\uparrow| + c_1\,|\downarrow\rangle\langle\downarrow|\right)^{\otimes N} \\ &= \sum_{m=0}^{N} c_0^{N-m} c_1^m \sum_{j=1}^{C_N^m} \Pi_j \left[\left(|\uparrow\rangle\langle\uparrow|\right)^{\otimes N-m}\left(|\downarrow\rangle\langle\downarrow|\right)^{\otimes m}\right],\end{aligned} \tag{4}$$

where Π_j denotes j-th permutation operator, $\Pi_j \in S_N$, and the number of the permutation operators is C_N^m. Inserting identity $I = \sum_{m=0}^{N} \sum_{\alpha=0}^{C_N^m - 1} |(N-m)\uparrow, m\downarrow\rangle_{\alpha\,\alpha}\langle(N-m)\uparrow, m\downarrow|$ in Eq.(4), we obtain

$$\begin{aligned}\rho^{\otimes N} = & \sum_{m=0}^{N} c_0^{N-m} c_1^m \sum_{m'=0}^{N} \sum_{\alpha=0}^{C_N^{m'}-1} \sum_{m''=0}^{N} \sum_{\alpha'=0}^{C_N^{m''}-1} |(N-m')\uparrow, m'\downarrow\rangle_{\alpha\,\alpha'}\langle(N-m'')\uparrow, m''\downarrow| \\ & {}_{\alpha}\langle(N-m')\uparrow, m'\downarrow| \sum_{j=1}^{C_N^m} \Pi_j \left[\left(|\uparrow\rangle\langle\uparrow|\right)^{\otimes N-m}\left(|\downarrow\rangle\langle\downarrow|\right)^{\otimes m}\right] |(N-m'')\uparrow, m''\downarrow\rangle_{\alpha'}\,.\end{aligned} \tag{5}$$

Here the orthonormal basis vector $|(N-m)\uparrow, m\downarrow\rangle_\alpha$ is defined in terms of the states with $(N-m)$ spins up and m spins down, which is like a symmetric state but with a phase in each term. The explicit expression is

$$\begin{aligned}&|(N-m)\uparrow, m\downarrow\rangle_\alpha \\ \equiv\ & \frac{1}{\sqrt{C_N^m}} \sum_{j=1}^{C_N^m} e^{2\pi i\alpha(j-1)/C_N^m} \Pi_j \left(|\uparrow\rangle^{\otimes N-m} |\downarrow\rangle^{\otimes m}\right).\end{aligned} \tag{6}$$

When $\alpha = 0$ the quantum state $|(N-m)\uparrow, m\downarrow\rangle_0 \equiv |(N-m)\uparrow, m\downarrow\rangle$ is symmetric state. Otherwise, the quantum state $|(N-m)\uparrow, m\downarrow\rangle_\alpha$ is asymmetric because of different phases in each item. For example, let $N = 3$ and $m = 1$, then $\alpha = 0, 1, 2$. We have $|2\uparrow,\downarrow\rangle_0 \equiv |2\uparrow,\downarrow\rangle = \frac{1}{\sqrt{3}}(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle)$, $|2\uparrow,\downarrow\rangle_1 = \frac{1}{\sqrt{3}}\left(|\uparrow\uparrow\downarrow\rangle + \omega\,|\uparrow\downarrow\uparrow\rangle + \omega^2\,|\downarrow\uparrow\uparrow\rangle\right)$, $|2\uparrow,\downarrow\rangle_2 = \frac{1}{\sqrt{3}}\left(|\uparrow\uparrow\downarrow\rangle + \omega^2\,|\uparrow\downarrow\uparrow\rangle + \omega\,|\downarrow\uparrow\uparrow\rangle\right)$, where $\omega = e^{2\pi i/3}$. It is obvious that $|2\uparrow,\downarrow\rangle_1$ and $|2\uparrow,\downarrow\rangle_2$ are not symmetric states under permutation operator.

It is easy to check that Eq.(5) is non-zero if and only if $m = m' = m''$. In the subspace constructed of $N-m$ qubits in state $|\uparrow\rangle$ and m qubits in state $|\downarrow\rangle$, $\sum_{j=1}^{C_N^m}\Pi_j\left[(|\uparrow\rangle\langle\uparrow|)^{\otimes N-m}(|\downarrow\rangle\langle\downarrow|)^{\otimes m}\right]$ is an C_N^m-dimensional identity operator. Then we obtain

$$\begin{aligned}
& {}_\alpha\langle(N-m')\uparrow, m'\downarrow|\sum_{j=1}^{C_N^m}\Pi_j\left[(|\uparrow\rangle\langle\uparrow|)^{\otimes N-m}(|\downarrow\rangle\langle\downarrow|)^{\otimes m}\right]|(N-m'')\uparrow, m''\downarrow\rangle_{\alpha'} \\
= {}& \delta_{m'm}\delta_{mm''}\,{}_\alpha\langle(N-m')\uparrow, m'\downarrow|\,(N-m'')\uparrow, m''\downarrow\rangle_{\alpha'} \\
= {}& \delta_{m'm}\delta_{mm''}\delta_{\alpha\alpha'}.
\end{aligned} \tag{7}$$

Substituting Eq.(7) into Eq.(5), we obtain

$$\rho^{\otimes N} = \sum_{m=0}^{N} c_0^{N-m} c_1^m \sum_{\alpha=0}^{C_N^m - 1} |(N-m)\uparrow, m\downarrow\rangle_{\alpha\alpha}\langle(N-m)\uparrow, m\downarrow|. \tag{8}$$

Eq.(8) is the final decomposition result, in which the state of N identical mixed qubit $\rho^{\otimes N}$ is decomposed into the sum of pure state density operators $|(N-m)\uparrow, m\downarrow\rangle_{\alpha\,\alpha}\langle(N-m)\uparrow, m\downarrow|$. One important property of this state is that each single-qubit reduced density operator is independent of the subscript α and is the same to each other, which is similar as a symmetric state.

In Ref.[18], the authors introduced another decomposition method,

$$\rho^{\otimes N} = \sum_{j=\langle\langle \frac{N}{2}\rangle\rangle}^{\frac{N}{2}} \sum_{m=-j}^{j} \sum_{\alpha=1}^{d_j} c_0^{\frac{N}{2}-m} c_1^{\frac{N}{2}+m} |jm\alpha\rangle\langle jm\alpha|, \tag{9}$$

where $\langle\langle\frac{N}{2}\rangle\rangle$ is 0 for N even, 1/2 for N odd, the detailed definition of the notations in the above formula will be presented in section VI. They also proposed a broadcasting procedure for this state. The scheme is like the following: By unitary transformation $U_{j,\alpha}^\dagger|jm\alpha\rangle$, we obtain $|jm\rangle \otimes \widetilde{|\uparrow,\downarrow\rangle}^{\otimes\frac{N}{2}-j}$, where $\widetilde{|\uparrow,\downarrow\rangle} = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ is the singlet state. Since the singlet state $\widetilde{|\uparrow,\downarrow\rangle}$ carries no information about ρ, the last $N-2j$ qubits are discarded. Then the rest $2j$ qubits in state $|jm\rangle$ are cloned by using the known UQCM of pure states. As a whole, the input states have different number of qubits and the number of output qubits is the same. One may find that this quantum broadcasting may not be universal. It is shown that the mixed states cannot be broadcasted equally well as the pure states.

We will use a different scheme to study the broadcasting of mixed states. For example, if we use the pure state decomposition presented in Ref.[18], we will broadcast the whole input state $|jm\alpha\rangle$ to M qubits. So the broadcasting procession does not depend on the specified form of the input, that is, we do not consider which part of the input should be cloned and which part should be discarded. Thus the scalar relation for quantum broadcasting still holds,

$$\rho_{out}^{single} = \eta\rho + \frac{1-\eta}{2}I,, \tag{10}$$

as mentioned previously, where η is the shrinking factor which is independent of input state ρ, and I is the identity operator.

As we know, the decomposition of N identical mixed states $\rho^{\otimes N}$ contains not only symmetric states but also asymmetric states. For the symmetric states input, there have been optimal universal cloning machine. However, the optimal universal quantum broadcasting of asymmetric states has not been proposed. In the following sections, by using our decomposition introduced above, we will demonstrate how to broadcast the states $|(N-m)\uparrow, m\downarrow\rangle_\alpha$ when $\alpha \neq 0$. Thus we can realize the broadcasting of mixed states ρ from N to $M \geq N$ copies. And we will also show that this broadcasting is optimal and universal. To show explicit how we broadcast the mixed qubits, we present in detail the broadcasting transformations for the 3 to M case.

III. $3 \longrightarrow M$ QUBITS QUANTUM BROADCASTING OF MIXED STATES

The result of 2 to M optimal universal quantum broadcasting of mixed states have been studied by Fan *et al* in Ref.[22]. In this section, we demonstrate in detail how to realize the optimal universal broadcasting from 3 to M copies of mixed states.

As we shown, an arbitrary mixed state of two-dimensional system takes the form

$$\rho = c_0 \left|\uparrow\right\rangle \left\langle\uparrow\right| + c_1 \left|\downarrow\right\rangle \left\langle\downarrow\right|, \tag{11}$$

where $c_0 + c_1 = 1$. According to Eq.(8), the three identical mixed states can be decomposed as

$$\begin{aligned}\rho^{\otimes 3} &= c_0^3 \left|3\uparrow\right\rangle \left\langle 3\uparrow\right| + c_0^2 c_1 \left|2\uparrow,\downarrow\right\rangle \left\langle 2\uparrow,\downarrow\right| \\ &+ c_0 c_1^2 \left|\uparrow, 2\downarrow\right\rangle \left\langle\uparrow, 2\downarrow\right| + c_1^3 \left|3\downarrow\right\rangle \left\langle 3\downarrow\right| \\ &+ c_0^2 c_1 \left|2\uparrow,\downarrow\right\rangle_{1\,1} \left\langle 2\uparrow,\downarrow\right| + c_0 c_1^2 \left|\uparrow, 2\downarrow\right\rangle_{1\,1} \left\langle\uparrow, 2\downarrow\right| \\ &+ c_0^2 c_1 \left|2\uparrow,\downarrow\right\rangle_{2\,2} \left\langle 2\uparrow,\downarrow\right| + c_0 c_1^2 \left|\uparrow, 2\downarrow\right\rangle_{2\,2} \left\langle\uparrow, 2\downarrow\right|.\end{aligned} \tag{12}$$

It is well known that the quantum cloning transformations of symmetric states are described as [11],

$$U_{3M} \left|3\uparrow\right\rangle \otimes R = \sum_{k=0}^{M-3} \beta_{0k} \left|(M-k)\uparrow, k\downarrow\right\rangle \otimes R_{\left|(M-3-k)\uparrow, k\downarrow\right\rangle}, \tag{13a}$$

$$U_{3M} \left|2\uparrow,\downarrow\right\rangle \otimes R = \sum_{k=0}^{M-3} \beta_{1k} \left|(M-1-k)\uparrow, (k+1)\downarrow\right\rangle \otimes R_{\left|(M-3-k)\uparrow, k\downarrow\right\rangle}, \tag{13b}$$

$$U_{3M} \left|\uparrow, 2\downarrow\right\rangle \otimes R = \sum_{k=0}^{M-3} \beta_{2k} \left|(M-2-k)\uparrow, (k+2)\downarrow\right\rangle \otimes R_{\left|(M-3-k)\uparrow, k\downarrow\right\rangle}, \tag{13c}$$

$$U_{3M} \left|3\downarrow\right\rangle \otimes R = \sum_{k=0}^{M-3} \beta_{3k} \left|(M-3-k)\uparrow, (k+3)\downarrow\right\rangle \otimes R_{\left|(M-3-k)\uparrow, k\downarrow\right\rangle}. \tag{13d}$$

where R denotes the blank state and the ancillary state, $R_{|\psi\rangle}$ denotes the ancillary state which should be traced out to obtain the output of the quantum broadcasting, in principle, the ancillary state is the same as the state $|\psi\rangle$. We write it as the subscript to distinguish the ancillary state from the copied state. By the cloning transformations Eq.(13a)-Eq.(13d), the output qubits are the same because the output states are M-qubit symmetric states.

As for the states with $\alpha \neq 0$, we propose the broadcasting transformations as below,

$$U_{3M} \left|2\uparrow,\downarrow\right\rangle_1 \otimes R = \sum_{k=0}^{M-3} \beta_{1k} \left|(M-1-k)\uparrow, (k+1)\downarrow\right\rangle_1 \otimes R_{\left|(M-3-k)\uparrow, k\downarrow\right\rangle_1}, \tag{13e}$$

$$U_{3M} \left|\uparrow, 2\downarrow\right\rangle_1 \otimes R = \sum_{k=0}^{M-3} \beta_{2k} \left|(M-2-k)\uparrow, (k+2)\downarrow\right\rangle_1 \otimes R_{\left|(M-3-k)\uparrow, k\downarrow\right\rangle_1}, \tag{13f}$$

$$U_{3M} \left|2\uparrow,\downarrow\right\rangle_2 \otimes R = \sum_{k=0}^{M-3} \beta_{1k} \left|(M-1-k)\uparrow, (k+1)\downarrow\right\rangle_2 \otimes R_{\left|(M-3-k)\uparrow, k\downarrow\right\rangle_2}, \tag{13g}$$

$$U_{3M} \left|\uparrow, 2\downarrow\right\rangle_2 \otimes R = \sum_{k=0}^{M-3} \beta_{2k} \left|(M-2-k)\uparrow, (k+2)\downarrow\right\rangle_2 \otimes R_{\left|(M-3-k)\uparrow, k\downarrow\right\rangle_2}, \tag{13h}$$

where, as we mentioned, $R_{|(M-3-k)\uparrow,k\downarrow\rangle_\alpha}$ $(\alpha = 0,1,2)$ are ancillary states and orthogonal to each other. A simple realization of them is that $R_{|(M-3-k)\uparrow,k\downarrow\rangle_\alpha} = |(M-3-k)\uparrow, k\downarrow\rangle_\alpha$. β_{mk} is the proposed amplitude for each item, and has the following form,

$$\beta_{mk} = \sqrt{\frac{(M-3)!\,(3+1)!}{(M+1)!}}\sqrt{\frac{(M-m-k)!}{(3-m)!\,(M-3-k)!}}\sqrt{\frac{(m+k)!}{m!k!}}, \tag{14}$$

where $m = 0,1,2,3$. The reason why we propose this kind of broadcasting transformation is that we would like to let input states $|(N-m)\uparrow, m\downarrow\rangle_\alpha$ have the same output reduced density operators for different α. As we already know the result for $\alpha = 0$ in Ref.[11], naturally we propose the same β_{mk} for different α. We note that the three-qubit asymmetric states are transformed into M-qubit asymmetric states, which have the same subscripts after the broadcasting transformation. However, they have different phases by definition in each superposition item. It is easy to check that the relations in Eq.(13a)-(13h) satisfy the unitary condition. By tracing out ancillary states $R_{|\psi\rangle}$ denoted as Rs, we obtain the output state of M qubits as below.

$$\begin{aligned}
\rho_{out} &= Tr_{Rs}\left[U_{3M}\left(\rho^{\otimes 3}\otimes R\right)U_{3M}^{\dagger}\right] \\
&= c_0^3\sum_{k=0}^{M-3}\beta_{0k}^2\,|(M-k)\uparrow, k\downarrow\rangle\,\langle(M-k)\uparrow, k\downarrow| \\
&\quad +c_0^2c_1\sum_{k=0}^{M-3}\beta_{1k}^2\,|(M-1-k)\uparrow,(k+1)\downarrow\rangle\,\langle(M-1-k)\uparrow,(k+1)\downarrow| \\
&\quad +c_0c_1^2\sum_{k=0}^{M-3}\beta_{2k}^2\,|(M-2-k)\uparrow,(k+2)\downarrow\rangle\,\langle(M-2-k)\uparrow,(k+2)\downarrow| \\
&\quad +c_1^3\sum_{k=0}^{M-3}\beta_{3k}\,|(M-3-k)\uparrow,(k+3)\downarrow\rangle\,\langle(M-3-k)\uparrow,(k+3)\downarrow| \\
&\quad +c_0^2c_1\sum_{k=0}^{M-3}\beta_{1k}^2\,|(M-1-k)\uparrow,(k+1)\downarrow\rangle_{11}\,\langle(M-1-k)\uparrow,(k+1)\downarrow| \\
&\quad +c_0c_1^2\sum_{k=0}^{M-3}\beta_{2k}^2\,|(M-2-k)\uparrow,(k+2)\downarrow\rangle_{11}\,\langle(M-2-k)\uparrow,(k+2)\downarrow| \\
&\quad +c_0^2c_1\sum_{k=0}^{M-3}\beta_{1k}^2\,|(M-1-k)\uparrow,(k+1)\downarrow\rangle_{22}\,\langle(M-1-k)\uparrow,(k+1)\downarrow| \\
&\quad +c_0c_1^2\sum_{k=0}^{M-3}\beta_{2k}^2\,|(M-2-k)\uparrow,(k+2)\downarrow\rangle_{22}\,\langle(M-2-k)\uparrow,(k+2)\downarrow|\,.
\end{aligned} \tag{15}$$

To evaluate the quality of output qubits, we should compare the single-qubit reduced density operator of output state with the input state ρ. As we know that in our decomposition, for a given multi-qubit state, symmetric or asymmetric (in a specified form as we defined), the single-qubit reduced density operators are the same to each other and are unrelated with symmetry. Then we have the following relations,

$$\begin{aligned}
&Tr_{M-1}\left[|(M-m-k)\uparrow,(m+k)\downarrow\rangle\,\langle(M-m-k)\uparrow,(m+k)\downarrow|\right] \\
&= \frac{C_{M-1}^{m+k}}{C_M^{m+k}}\,|\uparrow\rangle\,\langle\uparrow| + \frac{C_{M-1}^{m+k-1}}{C_M^{m+k}}\,|\downarrow\rangle\,\langle\downarrow| \\
&= \frac{M-m-k}{M}\,|\uparrow\rangle\,\langle\uparrow| + \frac{m+k}{M}\,|\downarrow\rangle\,\langle\downarrow|\,, \qquad (m=0,1,2,3)
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&Tr_{M-1}\left[|(M-m-k)\uparrow,(m+k)\downarrow\rangle\,\langle(M-m-k)\uparrow,(m+k)\downarrow|\right] \\
&= Tr_{M-1}\left[|(M-m-k)\uparrow,(m+k)\downarrow\rangle_{\alpha\alpha}\,\langle(M-m-k)\uparrow,(m+k)\downarrow|\right],
\end{aligned} \tag{17}$$

where $m = 1, 2$ and $\alpha = 1, 2$. Using Eq.(16) and Eq.(17), we obtain the scalar relation for single-qubit reduced density operator,

$$\rho_{out}^{single} = Tr_{M-1}\left(\rho_{out}\right) = \frac{3\left(M+2\right)}{5M}\rho + \frac{M-3}{5M}I. \tag{18}$$

It is obvious that the shrinking factor is independent of input state and reaches the upper bound $\frac{3(M+2)}{5M}$. So, we find the $3 \rightarrow M$ quantum broadcasting of mixed states which is universal, symmetric and optimal.

IV. GENERALIZATION: $N \longrightarrow M$ QUBITS QUANTUM BROADCASTING OF MIXED STATES

Here, we consider the general broadcasting case, that is, creating M qubits from N identical mixed states ($M \geq N$). Let the input state be N identical qubits, each in an unknown state described by a density operator ρ. According to Eq.(8), the decomposition of $\rho^{\otimes N}$ reads

$$\rho^{\otimes N} = \sum_{m=0}^{N} c_0^{N-m} c_1^m \sum_{\alpha=0}^{C_N^m - 1} \left|(N-m)\uparrow, m\downarrow\right\rangle_{\alpha\alpha} \left\langle (N-m)\uparrow, m\downarrow\right|. \tag{19}$$

We propose the general quantum broadcasting transformation from N to M qubits as follows,

$$\begin{aligned} & U_{NM}\left[\left|(N-m)\uparrow, m\downarrow\right\rangle_\alpha \otimes R\right] \\ = & \sum_{k=0}^{M-N} \beta_{mk}\left|(M-m-k)\uparrow, (m+k)\downarrow\right\rangle_\alpha \otimes R_{\left|(M-N-k)\uparrow, k\downarrow\right\rangle_\alpha}, \end{aligned} \tag{20}$$

where

$$\beta_{mk} = \sqrt{\frac{(M-N)!\,(N+1)!}{(M+1)!}}\sqrt{\frac{(M-m-k)!}{(N-m)!\,(M-N-k)!}}\sqrt{\frac{(m+k)!}{m!k!}}. \tag{21}$$

The unitary operator U_{NM} denotes the map from N identical qubits to M qubits. β_{mk} are amplitudes for each item. In Eq.(20), we still let the input state and output state have the same subscript. From Eq.(6), one can find their specific expressions.

Our aim is to show that the proposed broadcasting transformation is universal and optimal. So we should show that the scalar relation is satisfied and in it the shrinking factor does not depend on the input, that means this broadcasting is universal. At the same time, if the shrinking factor saturates its upper bound, we mean that this broadcasting procession is optimal.

By tracing out R, we obtain the single qubit output density operator

$$\begin{aligned} \rho_{out} &= Tr_R\left[U_{NM}\left(\rho^{\otimes N} \otimes R\right) U_{NM}^\dagger\right] \\ &= \sum_{m=0}^{N} c_0^{N-m} c_1^m \sum_{\alpha=0}^{C_N^m-1} Tr_R\left\{U_{NM}\left[\left|(N-m)\uparrow, m\downarrow\right\rangle_{\alpha\alpha}\left\langle (N-m)\uparrow, m\downarrow\right| \otimes R\right] U_{NM}^\dagger\right\} \\ &= \sum_{m=0}^{N} c_0^{N-m} c_1^m \sum_{\alpha=0}^{C_N^m-1} \sum_{k=0}^{M-N} \beta_{mk}^2 \left|(M-m-k)\uparrow, (m+k)\downarrow\right\rangle_{\alpha\,\alpha}\left\langle (M-m-k)\uparrow, (m+k)\downarrow\right|. \end{aligned} \tag{22}$$

Here the linearity superposition of quantum states is used. Then the output single-qubit reduced density operator reads

$$\begin{aligned} \rho_{out}^{single} &= Tr_{M-1}\left(\rho_{out}\right) \\ &= \sum_{m=0}^{N} c_0^{N-m} c_1^m \sum_{\alpha=0}^{C_N^m-1} \sum_{k=0}^{M-N} \beta_{mk}^2 Tr_{M-1}\left[\left|(M-m-k)\uparrow, (m+k)\downarrow\right\rangle_{\alpha\,\alpha}\left\langle (M-m-k)\uparrow, (m+k)\downarrow\right|\right] \\ &= \sum_{m=0}^{N} c_0^{N-m} c_1^m \sum_{\alpha=0}^{C_N^m-1} \sum_{k=0}^{M-N} \beta_{mk}^2 \left(\frac{M-m-k}{M}\left|\uparrow\right\rangle\left\langle\uparrow\right| + \frac{m+k}{M}\left|\downarrow\right\rangle\left\langle\downarrow\right|\right) \\ &= \frac{N\left(M+2\right)}{M\left(N+2\right)}\rho + \frac{M-N}{M\left(N+2\right)}I. \end{aligned} \tag{23}$$

To calculate Eq.(23), the following relations are used.

$$\begin{aligned}
& Tr_{M-1}\left[\left|(M-m-k)\uparrow,(m+k)\downarrow\right\rangle_{\alpha\alpha}\left\langle(M-m-k)\uparrow,(m+k)\downarrow\right|\right] \\
= & \frac{C_{M-1}^{m+k}}{C_M^{m+k}}\left|\uparrow\right\rangle\left\langle\uparrow\right| + \frac{C_{M-1}^{m+k-1}}{C_M^{m+k}}\left|\downarrow\right\rangle\left\langle\downarrow\right| \\
= & \frac{M-m-k}{M}\left|\uparrow\right\rangle\left\langle\uparrow\right| + \frac{m+k}{M}\left|\downarrow\right\rangle\left\langle\downarrow\right|.
\end{aligned} \tag{24}$$

Obviously scalar relation between single qubit input ρ and single-qubit output reduced density operator is satisfied, $\rho_{out}^{single}=\frac{N(M+2)}{M(N+2)}\rho+\frac{M-N}{M(N+2)}I$, and we find the shrinking factor $\frac{N(M+2)}{M(N+2)}$ achieves the well known upper bound. Thus our proposed quantum broadcasting is optimal. Since the upper bound is saturated, we know this broadcasting procession can copy mixed state equally well as the pure state. In short we demonstrated an optimal quantum broadcasting which can transform N identical mixed states into M-qubit states ($M \geqslant N$).

V. OTHER OPTIMAL UNIVERSAL QUANTUM BROADCASTING PROCEDURES FOR MIXED STATES

Besides the above mentioned quantum broadcasting, there are several other quantum broadcasting procedures which are still optimal. In Eq.(20), the N-qubit symmetric states are transformed to M-qubit symmetric states, while states $|(N-m)\uparrow,m\downarrow\rangle_\alpha$ are transformed to M-qubit states with similar form. In the process of this quantum broadcasting, the input state and output state have the same subscript α. However, we found that when $|(N-m)\uparrow,m\downarrow\rangle_\alpha$ is transformed into state $\frac{1}{\sqrt{C_M^{m+k}}}\sum_{\alpha=0}^{C_M^{m+k}-1}|(M-m-k)\uparrow,(m+k)\downarrow\rangle_\alpha$, it is also an optimal universal broadcasting transformation. The transformation is presented as follows,

$$\begin{aligned}
& U'_{NM}\left|(N-m)\uparrow,m\downarrow\right\rangle_\alpha\otimes R \\
= & \sum_{k=0}^{M-N}\beta_{mk}\frac{1}{\sqrt{C_M^{m+k}}}\sum_{\alpha'=0}^{C_M^{m+k}-1}\left|(M-m-k)\uparrow,(m+k)\downarrow\right\rangle_{\alpha'}\otimes R_{|(M-N-k)\uparrow,k\downarrow\rangle_{\alpha'}}.
\end{aligned} \tag{25}$$

where terms in the superposition of the output state are changed to another form compared with the results in Section IV.

If we use the decomposition method in Ref.[18], it is still possible to realize other two quantum broadcastings. The first one takes the form

$$U''_{NM}\left|jm\alpha\right\rangle\otimes R=\sum_{k=0}^{M-N}\Upsilon_{mk}\left|\left(M-\frac{N}{2}-m-k\right)\uparrow,\left(\frac{N}{2}+m+k\right)\downarrow\right\rangle_\alpha\otimes R_{|(M-N-k)\uparrow,k\downarrow\rangle_\alpha}. \tag{26}$$

The second one takes the form

$$\begin{aligned}
& U'''_{NM}\left|jm\alpha\right\rangle\otimes R \\
= & \sum_{k=0}^{M-N}\Upsilon_{mk}\frac{1}{\sqrt{C_M^{m+k}}}\sum_{\alpha'=0}^{C_M^{m+k}-1}\left|\left(M-\frac{N}{2}-m-k\right)\uparrow,\left(\frac{N}{2}+m+k\right)\downarrow\right\rangle_{\alpha'}\otimes R_{|(M-N-k)\uparrow,k\downarrow\rangle_{\alpha'}},
\end{aligned} \tag{27}$$

where the normalization factor

$$\Upsilon_{mk}=\sqrt{\frac{(M-N)!\,(N+1)!}{(M+1)!}}\sqrt{\frac{(M-\frac{N}{2}-m-k)!}{(\frac{N}{2}-m)!\,(M-N-k)!}}\sqrt{\frac{(\frac{N}{2}+m+k)!}{(\frac{N}{2}+m)!k!}}. \tag{28}$$

It is straightforward to check that the above quantum broadcasting procedures are universal, symmetric and optimal. That is to say, for N copies of given mixed state ρ there exist not only one optimal universal quantum broadcasting procedures.

VI. $N \longrightarrow M$ COPYING BY SUPERBROADCASTING SCHEME

Cirac *et al* proposed an identical mixed states purification scheme followed by a pure state cloning machine [18], and this is later related with the mixed states superbroadcasting [20, 21]. Here, we review those results and compare them with the results in this paper. Let the input state be N identical qubits, each in an unknown state described by a density operator $\rho = c_0 \left|\uparrow\right\rangle\left\langle\uparrow\right| + c_1 \left|\downarrow\right\rangle\left\langle\downarrow\right|$, where $c_0 + c_1 = 1$. According to Eq.(9), the decomposition of $\rho^{\otimes N}$ reads

$$\rho^{\otimes N} = \sum_{j=\langle\langle \frac{N}{2} \rangle\rangle}^{\frac{N}{2}} \sum_{m=-j}^{j} \sum_{\alpha=1}^{d_j} c_0^{\frac{N}{2}-m} c_1^{\frac{N}{2}+m} \left|jm\alpha\right\rangle\left\langle jm\alpha\right|, \tag{29}$$

where

$$d_j = C_N^{\frac{N}{2}-j} - C_N^{\frac{N}{2}-j-1}, \tag{30}$$

$$\left|jm\alpha\right\rangle = U_{j,\alpha}\left|jm1\right\rangle = U_{j,\alpha}\left[\left|(j-m)\uparrow,(j+m)\downarrow\right\rangle \otimes \widetilde{\left|\uparrow,\downarrow\right\rangle}^{\otimes \frac{N}{2}-j}\right]. \tag{31}$$

Let $\alpha = 0$ for $j = \frac{N}{2}$, and $\alpha = 1, 2, 3, ..., d_j$ when $j \neq \frac{N}{2}$.

The detailed super-broadcasting scheme is shown as follows. Firstly, we measure the input N qubits $\rho^{\otimes N}$, if getting state $\left|jm\alpha\right\rangle$, we take an unitary operation $U_{j,\alpha}$ and change it into state $\left|jm\right\rangle \otimes \widetilde{\left|\uparrow,\downarrow\right\rangle}^{\otimes \frac{N}{2}-j} = \left|(j-m)\uparrow,(j+m)\downarrow\right\rangle \otimes \widetilde{\left|\uparrow,\downarrow\right\rangle}^{\otimes \frac{N}{2}-j}$. Then we discard the last $N-2j$ qubits, because these qubits are singlets without any information of ρ. Secondly, we use the optimal universal cloning transformations of pure states to clone $2j$ qubits in symmetric state to M output qubits.

$$U_{2jM}\left|jm\right\rangle \otimes R = \sum_{k=0}^{M-2j} \beta_{mk}\left|(M-j-m-k)\uparrow,(j+m+k)\downarrow\right\rangle \otimes R_{(M-2j-k)\uparrow,k\downarrow}, \tag{32}$$

where

$$\beta_{mk} = \sqrt{\frac{(M-2j)!\,(2j+1)!}{(M+1)!}}\sqrt{\frac{(M-j-m-k)!}{(j-m)!\,(M-2j-k)!}\frac{(j+m+k)!}{(j+m)!k!}}. \tag{33}$$

The unitary operator U_{2jM} denotes the map from $2j$ qubits to M qubits. By tracing out ancillary states Rs, we obtain the output density operator

$$\begin{aligned}\rho_{out} &= Tr_{Rs}\left[\sum_{j=\langle\langle \frac{N}{2} \rangle\rangle}^{\frac{N}{2}} \sum_{m=-j}^{j} c_0^{\frac{N}{2}-m} c_1^{\frac{N}{2}+m} d_j U_{2jM}\left(\left|jm\right\rangle\left\langle jm\right| \otimes R\right) U_{2jM}^{\dagger}\right] \\ &= \sum_{j=\langle\langle \frac{N}{2} \rangle\rangle}^{\frac{N}{2}} \sum_{m=-j}^{j} c_0^{\frac{N}{2}-m} c_1^{\frac{N}{2}+m} d_j \sum_{k=0}^{M-2j} \beta_{mk}^2 \\ &\quad \times \left|(M-j-m-k)\uparrow,(j+m+k)\downarrow\right\rangle\left\langle(M-j-m-k)\uparrow,(j+m+k)\downarrow\right|.\end{aligned} \tag{34}$$

By using

$$\begin{aligned}&Tr_{M-1}\left[\left|(M-j-m-k)\uparrow,(j+m+k)\downarrow\right\rangle j\left\langle(M-j-m-k)\uparrow,(j+m+k)\downarrow\right|\right] \\ &= \frac{C_{M-1}^{j+m+k}}{C_M^{j+m+k}}\left|\uparrow\right\rangle\left\langle\uparrow\right| + \frac{C_{M-1}^{j+m+k-1}}{C_M^{j+m+k}}\left|\downarrow\right\rangle\left\langle\downarrow\right| \\ &= \frac{M-j-m-k}{M}\left|\uparrow\right\rangle\left\langle\uparrow\right| + \frac{j+m+k}{M}\left|\downarrow\right\rangle\left\langle\downarrow\right|,\end{aligned}$$

the output single-qubit reduced density operator reads

$$\begin{aligned}
\rho_{single}^{out} &= Tr_{M-1}\left(\rho_{out}\right) \\
&= \sum_{j=\langle\langle\frac{N}{2}\rangle\rangle}^{\frac{N}{2}} \sum_{m=-j}^{j} c_0^{\frac{N}{2}-m} c_1^{\frac{N}{2}+m} d_j \sum_{k=0}^{M-2j} \beta_{mk}^2 \left(\frac{M-j-m-k}{M} \left|\uparrow\right\rangle\left\langle\uparrow\right| + \frac{j+m+k}{M} \left|\downarrow\right\rangle\left\langle\downarrow\right| \right) \\
&= c_0'' \left|\uparrow\right\rangle\left\langle\uparrow\right| + c_1'' \left|\downarrow\right\rangle\left\langle\downarrow\right| \\
&= \frac{1}{2}\left(I + r'' \overrightarrow{n}\overrightarrow{\sigma}\right) \\
&= r'' \left|\uparrow\right\rangle\left\langle\uparrow\right| + \frac{1-r''}{2} I,
\end{aligned} \tag{35}$$

where

$$c_0'' = \sum_{j=\langle\langle\frac{N}{2}\rangle\rangle}^{\frac{N}{2}} \sum_{m=-j}^{j} c_0^{\frac{N}{2}-m} c_1^{\frac{N}{2}+m} d_j \sum_{k=0}^{M-2j} \beta_{mk}^2 \frac{M-j-m-k}{M}, \tag{36}$$

$$c_1'' = \sum_{j=\langle\langle\frac{N}{2}\rangle\rangle}^{\frac{N}{2}} \sum_{m=-j}^{j} c_0^{\frac{N}{2}-m} c_1^{\frac{N}{2}+m} d_j \sum_{k=0}^{M-2j} \beta_{mk}^2 \frac{j+m+k}{M}, \tag{37}$$

$$\begin{aligned}
r'' &= c_0'' - c_1'' \\
&= \sum_{j=\langle\langle\frac{N}{2}\rangle\rangle}^{\frac{N}{2}} \sum_{m=-j}^{j} c_0^{\frac{N}{2}-m} c_1^{\frac{N}{2}+m} d_j \sum_{k=0}^{M-2j} \beta_{mk}^2 \frac{M-2\left(j+m+k\right)}{M} \\
&= \sum_{j=\langle\langle\frac{N}{2}\rangle\rangle}^{\frac{N}{2}} \sum_{m=-j}^{j} c_0^{\frac{N}{2}-m} c_1^{\frac{N}{2}+m} d_j \left[\frac{-\left(M+2\right)m}{M\left(j+1\right)}\right] \\
&= -\frac{M+2}{M} \sum_{j=\langle\langle\frac{N}{2}\rangle\rangle}^{\frac{N}{2}} \frac{d_j}{(j+1)} \sum_{m=-j}^{j} \left(\frac{1+r}{2}\right)^{\frac{N}{2}-m} \left(\frac{1-r}{2}\right)^{\frac{N}{2}+m} m.
\end{aligned} \tag{38}$$

Obviously the output single-qubit reduced density operator takes the form

$$\rho_{single}^{out} = \frac{r''}{r}\rho + \frac{1-r''/r}{2} I, \tag{39}$$

where the shrinking factor is $\eta = \frac{r''}{r}$, and we have

$$\eta = -\frac{M+2}{rM} \sum_{j=\langle\langle\frac{N}{2}\rangle\rangle}^{\frac{N}{2}} \frac{d_j}{(j+1)} \sum_{m=-j}^{j} m \left(\frac{1+r}{2}\right)^{\frac{N}{2}-m} \left(\frac{1-r}{2}\right)^{\frac{N}{2}+m}, \tag{40}$$

which depends on r, i.e., a parameter depending on the input state ρ. When $r = 1$, the input state is pure and the shrinking factor reaches the optimal bound $\eta = \frac{N(M+2)}{M(N+2)}$. While when $r \neq 1$, we find $\eta > \frac{N(M+2)}{M(N+2)}$, that is to say, for mixed state broadcasting the shrinking factor can larger than that of optimal pure state cloning, though which *does* depend on the purity of input mixed states.

VII. CONCLUSION

We introduce a pure states decomposition for N identical mixed states $\rho^{\otimes N}$. An optimal quantum broadcasting is proposed to copy the mixed states input. We show that the mixed states can be quantum broadcasted equally well

as that of the pure states in the sense that the shrinking factors are the same for both cases. The shrinking factor is a constant and is thus independent of each input qubit ρ, and in this sense we say it is universal. This broadcasting procedure is optimal since the shrinking factor is optimal. The optimal broadcasting procession for mixed states is not unique, and we also present other different quantum broadcasting procedures with similar properties, i.e.,universal and optimal.

In the broadcasting procession, we assume that each input qubit is an arbitrary mixed state ρ, no prior information is available. In case that partial information of the initial state is known, it is generally expected that a higher shrinking factor can be achieved. Such as the super-broadcasting scheme [18, 20, 21], a higher shrinking factor can be achieved which depends on the purity of the mixed input state. For the cloning scheme proposed in this paper, we restrict that the input state can be arbitrary, pure or mixed, and no prior information is available. Still we let the shrinking factor be *universal*. Thus our quantum broadcasting provides a unified scheme to broadcast mixed states and pure states.

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[1] W. K. Wootters and W. H. Zurek, Nature 299, 802 (1982)
[2] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa, and B. Schumacher, Phys. Rev. Lett. 76, 2818 (1996)
[3] V. Bužek and M. Hillery, Phys. Rev. A 54, 1844 (1996)
[4] V. Bužek and M. Hillery, Phys. Rev. Lett. 81, 5003 (1998)
[5] L.M. Duan, and G.C. Guo, Phys. Rev. Lett. 80, 4999 (1998).
[6] N. Gisin and S. Massar, Phys. Rev. Lett. 79, 2153 (1997)
[7] D. Bruss, A. Ekert, and C. Macchiavello, Phys. Rev. Lett. 81, 2598 (1998)
[8] D. Bruss, D. P. DiVincenzo, A. Ekert, C. A. Fuchs, C. Macchiavello, and J. Smolin, Phys. Rev. A 57, 2368 (1998)
[9] R. F. Werner, Phys. Rev. A 58, 1827 (1998)
[10] P. Zanardi, Phys. Rev. A 58, 3484 (1998)
[11] H. Fan, K. Matsumoto and M. Wadati, Phys. Rev. A 64, 064301 (2001)
[12] N.J. Cerf, Phys. Rev. Lett. 84, 4497 (2000)
[13] S. Iblisdir, A. Acin, N. J. Cerf, R.Filip, J. Fiurasek, and N. Gisin, Phys. Rev. A 72, 042328 (2005)
[14] H. Fan, Phys. Rev. A 68, 054301 (2003)
[15] V. Scarani, S. Iblisdir, N. Gisin, and A. Acin, Rev. Mod. Phys. **77**, 1225 (2006).
[16] N. J. Cerf, and J. Fiurasek, *Progress in Optics*, **49**, 455 (Elsevier 2006).
[17] H. Fan, Topics in Applied Physics **102**, 63 (2006).
[18] J. I. Cirac, A. K. Ekert and C. Macchiavello, Phys. Rev. Lett. 82, 4344 (1999)
[19] M. Keyl and R. F. Werner, Annales Henri Poincare, **2**, 1 (2001).
[20] G. M. D'Ariano, C. Macchiavello, and P. Perinotti, Phys. Rev. Lett. 95, 060503 (2005)
[21] F. Buscemi, G. M. D'Ariano, C. Macchiavello, and P. Perinotti, quant-ph/0510155
[22] H. Fan, B. Y. Liu and K. J. Shi, Quant. Info. Comput. **7**, 551 (20007).
[23] L. Chen, and Y. X. Chen, *Mixed qubit cannot be universally broadcast*, quant-ph/0612205.